Parsimonious HJM-FMM Model with the New Risk-Free Term Rates

Financial Engineering Department

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Abstract

After the crisis that has affected financial markets in 2007-2009, the Financial Stability Board (FSB) initiated a process of reform for the principal interest rate benchmarks. This operation aimed to address concerns about the robustness and integrity of the existing benchmarks, the Interbank Offered Rates (IBORs), such as the London Interbank Offered Rate (LIBOR), which had been found to be vulnerable to manipulation. As part of this process, the FSB introduced new Risk-Free term Rates (RFRs) as alternative benchmarks to LIBOR. These new RFRs, such as the Overnight Indexed Swap rate (OIS), are based on transactions between market participants, rather than on banks' submissions, making them less susceptible to manipulation. The main difference between the old and the new rates is that the former are term rates while the latter have a Overnight (O/N) maturity. However, the possibility of building a term structure also for the RFRs made it possible to develop an extension of the traditional LIBOR Market Model (LMM), which is strongly related to the old interest rates, to a new framework designated as Forward Market Model (FMM). Such model is able to simultaneously describe the evolution of both the forward-looking (LIBOR-like) and the new backward-looking (setting-in-arrears) term rates using the same stochastic process for both. It is due to note that, when switching from forward-looking to backward-looking term rates, the properties of the standard interest-rate modeling framework are not only maintained but also enriched. Setting-in-arrears rates, in fact, own all the relevant analytical features held by the IBORs, such as the martingale property under the related forward measure, plus other nice qualities, such as a simple analytic formula for the drift under the risk-neutral measure. Hence, they show to be a very valid replacement for the old term rates from an analytical point of view. Our research paper presents a novel Heath-Jarrow-Morton (HJM) model that exploits the potentialities of these new risk-free term rates. Clearly, the HJM framework is a powerful tool for pricing derivatives, but it has its limitations. The new RFRs present a unique challenge, as they require a model that can take into account the specific characteristics of these new benchmarks. Our HJM model uses a parsimonious (finite-dimensional) Markovian framework, with a separable volatility form, that generates the dynamics of the extended forward rates, that are equivalent to the FMM ones. Such model exploits a reduced number of free parameters compared to the complete framework of Lyashenko and Mercurio [1], [2], thus demanding a much lower computational effort. Moreover, it constitutes a single-curve framework where all the structures are generated starting from a single rate, the RFR, thus differing from the previous multi-curve model of Moreni and Pallavicini [3] based on the old IBOR rates. We would like to remark here that this model, which combines precision and flexibility, is specifically designed to take advantage of the new RFRs, making it the go-to choice for pricing vanilla derivatives on the new O/N interest rate benchmarks. Indeed, due to its ability to handle the complexities of the new rates, while maintaining a high level of accuracy, it is well suited to meet the needs of today's market participants, which are seeking a more robust, efficient, and reliable way to achieve this goal. Specifically, in this paper, after a brief introduction on the fundamental definitions of the FMM, we draw final expressions for the risk-free forward rate dynamics and the (approximated) risk-free swap rate dynamics as well as valuation Black-like formulas for derivatives on these rates (caps and European swaptions), by adopting a specific model realization with a deterministic volatility. It is worth noticing that, thanks to the concept of extended zero-coupon bond, on which our FMM formulation is based, the forward and swap rate dynamics are defined for all times, even those beyond their natural expiries. As an additional thing, by restricting ourselves to a HJM two-factor model and following [4] and [5], we derive explicit pricing formulas for European (payer and receiver) swaptions.

1 Introduction

Starting from 2013, a fundamental process of reform of the principal interest rate benchmarks in the money market has been conducted by the Financial Stability Board (FSB), with the intent of replacing the Interbank Offered Rates (IBORs) with alternative nearly Risk-Free Rates (RFRs). Following the credit crisis of 2007-2009, in fact, the significant decrease in the number of trades in the interbank market has led to several attempts by banks to manipulate the IBORs. Therefore, since they possess the property of being more resilient to manipulation, the new RFRs appear to be better suited as the reference rates for certain financial transactions (for a detailed overview of the matter see [6], [7], [8]).
However, while the IBORs are term rates, the RFRs are of overnight (O/N) type, and thus, in order to exploit the latter as a replacement of the former, we must build a term structure also for the RFRs, which is always possible by adopting one of the two approaches detailed below [9], [10]:

1. use of a daily compounded setting-in-arrears rate (starting from the overnight reference rate), which is backward-looking by definition, i.e. known only at the end of the corresponding application period;

2. use of the market to predict the rate introduced above, which produces by definition a forward-looking rate, that is, a rate whose value is known at the beginning of the related application period.

With a term structure built in this way, it is then possible to simulate both forward-looking and backward-looking term rates using a unique stochastic process for both. The joint modeling of the two rates leads to an extension of the classic single-curve LIBOR Market Model (LMM) which is called the generalized Forward Market Model (FMM) [1], [2], [11]. Therefore, the FMM turns out to be a more complete model with respect to the LMM because, while preserving the dynamics of the forward-looking (LIBOR-like) rates, it supplies extra information, such as the rate dynamics under the classic money-market risk-neutral measure, and not only under the discrete spot measure as in the traditional LMM. Furthermore, the FMM framework is based on the concept of extended zero-coupon bonds, which we will describe in detail in Section 2.

Now, let us introduce the Heath-Jarrow-Morton (HJM) model [12]. This model has been developed as an alternative to short-rate models, with the aim of creating a quite general framework to describe the dynamics of interest rates. In particular, it constitutes an arbitrage-free framework to outline the stochastic evolution of the whole yield curve, through the use of the instantaneous forward rates as fundamental quantities to model. It is due to say that, in contrast to spot rates, for which it is possible to choose an arbitrary dynamics, the drift of the forward rates evolution process has to be fully specified by the instantaneous volatility coefficient. Also, this volatility function cannot be any, but only a restricted class of volatilities ensures a Markovian dynamics of the interest rates.

Our goal is to construct a parsimonious (finite-dimensional) Markovian HJM framework, with a separable volatility structure, which produces the dynamics of the extended forward rates, that are equivalent to the FMM ones. The reason why we chose to adopt a parsimonious model, thus deviating from the complete framework of the original papers of Lyashenko and Mercurio (2019a, 2019b) [1], [2], is the willing to overcome the enormous computational effort caused by the huge amount of parameters the complete model has behind. Furthermore, our HJM model exploits the potentialities of the new risk-free term rates, and differs from the previous (multi-curve) model of Moreni and Pallavicini (2014) [3], built using the old IBOR rates, as it consists of a mono-curve framework where all the structures are generated starting from a single rate, the RFR. Furthermore, it demonstrates to be very well suited to price vanilla derivatives on the new O/N interest rate benchmarks.

The paper is structured as follows. Section 2 introduces the basic definitions and the fundamental concepts of the FMM that underlie the next treatment; Section 3 is devoted to the description of the extended HJM model, which includes the constraints adopted on the volatility process and the risk-neutral measure formulation of the risk-free forward rate and discount factor dynamics; Section 4 proposes a particular realization of the extended HJM model, based on a deterministic volatility, which is used to infer the specific dynamics of the risk-free forward and swap rates and evaluate RFR vanilla derivatives accordingly; Section 6 is dedicated to obtain explicit pricing formulas for the European swaptions in a HJM two-factor model, with, in addition, a peculiar formulation of them drawn from the assumptions made in Sec. 4; finally, Section 7 reviews the work done.

2 Basic definitions and notation of the FMM

In our work we assume a single-curve framework where the interest rates are risk-free. The *instantaneous risk-free rate* at time t is denoted by $r(t)$ and has an associated money-market (or bank) account $B(t)$ which accrues continuously starting from $B(0) = 1$, i.e.

$$\mathrm{d}B(t) = r(t)B(t)\mathrm{d}t\,, \tag{1}$$

and hence

$$B(t) = e^{\int_0^t r(u)\mathrm{d}u}\,. \tag{2}$$

Moreover, we assume that a risk-neutral measure Q exists with $B(t)$ as associated numeraire. Thus, the price at time t of the risk-free zero-coupon bond with maturity T is

$$P(t,T) = \mathbb{E}\big[e^{-\int_t^T r(u)\mathrm{d}u}|\mathcal{F}_t\big]\,, \tag{3}$$

where $\mathbb{E}$ indicates the expectation with respect to Q and $\mathcal{F}_t$ is the sigma-algebra generated by the model risk factors up to time t, that is, the information available in the market at t. Since it represents the value of a contract expiring at T, Eq. (3) turns out to be defined only for $t \leq T$. However, it is possible to extend it to times t such that $t \geq T$, in the following way:

$$P(t,T) = \mathbb{E}\big[e^{\int_T^t r(u)\mathrm{d}u}|\mathcal{F}_t\big] = e^{\int_T^t r(u)\mathrm{d}u} = \frac{B(t)}{B(T)}\,, \tag{4}$$

where we have used Eqs. (2) and (3) and the fact that the quantity $\int_T^t r(u)\mathrm{d}u$ is $\mathcal{F}_t$-measurable.

Now, if we consider a self-financing strategy that consists of buying the zero-coupon bond with maturity T and reinvesting the proceeds of the bond's unit notional received at T at the risk-free rate $r(t)$ from T onwards, we can write such strategy as follows:

$$Y_T(t) = \begin{cases} P(t,T) & \text{for } t \leq T \\ e^{\int_T^t r(u)\mathrm{d}u} = \frac{B(t)}{B(T)} & \text{for } t > T\,. \end{cases} \tag{5}$$

From this equation it is easy to see that $Y_T(t)$ coincides with the extended bond price defined in Eq. (3). Hence, for any given T, the equality $Y_T(t) = P(t,T)$ holds for all times t, and thus we will refer to $P(t,T)$ as the extended zero-coupon bond price at time t with maturity T. In particular, we have that $P(t,0) = B(t)$, i.e. the bank account is to all effects a zero-coupon bond with immediate expiry. Furthermore, if we define the *instantaneous risk-free forward* *rate* at time t with maturity T, $f(t,T)$, in such a way that $f(t,u) = r(u)$ for $t \geq u$, then the extended bond price reads

$$P(t,T) = e^{-\int_t^T f(t,u)\mathrm{d}u}\,, \tag{6}$$

in the same way as classic bond prices.

2.1 Extended T-forward measure

The T-forward measure is defined as the probability measure associated to a bond with maturity T, and is often referred to as Q^T. Its name derives from the fact that, under the forward measure, forward prices are martingales (for a detailed discussion, see [13] and [3]).

Similarly, we can define the extended T-forward measure as the equivalent martingale measure associated with the extended bond price $P(t,T)$, and we also call it Q^T. This definition is admissible because the extended zero-coupon bond is still a suitable numeraire, in that it represents the value of a self-financing strategy and is strictly positive. The extended T-forward measure combines the classic T-forward measure up to the maturity T and the risk-neutral bank account measure after the maturity T. Hence, contrarily to the classic Q^T, which is defined only for $t \leq T$, the extended Q^T is specified for all times t.

2.2 Risk-free term rates

In the following, we consider a time structure $0 = T_0, T_1, ..., T_M$, and denote the year fraction associated to the interval $[T_{j-1}, T_j)$ by τ_j, $j = 1, ..., M$. For each period $[T_{j-1}, T_j)$, we can approximate the daily-compounded *backward-looking rate* as follows:

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\mathrm{d}u} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(T_j)}{B(T_{j-1})} - 1\right] = \frac{1}{\tau_j}\left[P(T_j, T_{j-1}) - 1\right]. \tag{7}$$

This rate evolves within its corresponding accrual period and hence its fixing value will only be known at the end of this period. As another option, we can define a kind of rate which is fixed at the beginning of the accrual period, and thus maintains the same value throughout the corresponding interval. This is the fixed rate that has to be exchanged at time T_j for the forward bank account $B(T_j)/B(T_{j-1})$ in such a way that this swap is zero at time T_{j-1}: it is labeled as *forward-looking rate* and can be written as

$$F(T_{j-1}, T_j) = \mathbb{E}^{T_j}[R(T_{j-1}, T_j)|\mathcal{F}_{T_{j-1}}]\,, \tag{8}$$

where we have exploited the no-arbitrage rule. Furthermore, $\mathbb{E}^{T_j}$ represents the expectation with respect to the T_j-forward measure (whose associated numeraire is the T_j-maturity bond price) and $\mathcal{F}_{T_{j-1}}$ is the information available in the market at time T_{j-1}.

2.3 Backward-looking and forward-looking forward rates

Taking into account the time structure outlined in the above section, we can define, for each $j = 1, ..., M$, the time-t *backward-looking forward rate* as the fixed rate value K_R that nullifies at time t the swaplet paying $\tau_j[R(T_{j-1}, T_j) - K_R]$ at time T_j. This rate can be written as follows:

$$R(t, T_j) = \mathbb{E}^{T_j}[R(T_{j-1}, T_j)|\mathcal{F}_t] = \frac{1}{\tau_j}\left[\frac{P(t, T_{j-1})}{P(t, T_j)} - 1\right], \tag{9}$$

where the first equality holds by no-arbitrage while the second one has been obtained by changing the measure from Q^{T_j} to Q (see [1] for the details). Eq. (9) coincides with the classic simply-compounded forward rate formula, but differs from it in that it holds for all times t. Furthermore, $R(t, T_j)$ is a martingale under the T_j-forward measure and

(i) gives the fixing of the forward-looking spot rate at time T_{j-1}: $R(T_{j-1}, T_j) = F(T_{j-1}, T_j)$;

(ii) gives the fixing of the realized backward-looking rate at time T_j: $R(T_j, T_j) = R(T_{j-1}, T_j)$;

(iii) stops evolving after time T_j: $R(t, T_j) = R(T_{j-1}, T_j)$.

We also note that when $t \in (T_{j-1}, T_j)$, i.e. within the accrual period, the backward-looking forward rate can be rewritten as

$$R(t, T_j) = \frac{1}{\tau_j}\left[\frac{e^{\int_{T_{j-1}}^{t} r(u)\mathrm{d}u}}{P(t, T_j)} - 1\right] = \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{t} r(u)\mathrm{d}u + \int_{t}^{T_j} f(t,u)\mathrm{d}u} - 1\right], \tag{10}$$

that is, it aggregates values of realized risk-free rates $r(u)$, with $T_{j-1} < u < t$, and spot forward rates $f(t, u)$, with $t < u < T_j$.

Analogously to $R(t, T_j)$, we can define the time-t *forward-looking forward rate* as the fixed rate value K_F that nullifies at time t the swaplet paying $\tau_j[F(T_{j-1}, T_j) - K_F]$ at time T_j. Such rate, for $t \leq T_{j-1}$, reads, by no-arbitrage:

$$F(t, T_j) = \mathbb{E}^{T_j}[F(T_{j-1}, T_j)|\mathcal{F}_t] = R(t, T_j)\ , \tag{11}$$

where we have exploited the first equality of Eq. (9) and observation (i). For times $t > T_{j-1}$, the forward-looking forward rate remains fixed and equal to $F(T_{j-1}, T_j)$.

To conclude, we outline the evolution of the two forward rates for all times t, which is as follows:

- for $t \leq T_{j-1}$, the two rates are equal;
- for $t = T_{j-1}$, the forward-looking forward rate fixes and stops evolving, while the backward-looking forward rate continues to evolve until $t = T_j$;
- for $t \geq T_j$, the backward-looking forward rate keeps the value reached at T_j.

From such evolution we deduce that, for $j = 1, ..., M$, the backward-looking and the forward-looking forward rates can be expressed by a common rate, which with some abuse of notation we will designate by $R(t, T_j) \equiv R_j(t)$.

3 The Heath-Jarrow-Morton model with risk-free rates

In this section we will present an extension of the Heath-Jarrow-Morton model, which will be built as a single-curve framework that makes use of the potential of the new RFR rates. In particular, we will show the dynamics of the instantaneous risk-free forward rate and the extended zero-coupon bond both under the extended T-forward measure and in the risk-neutral measure formulation. Such dynamics will be rewritten by starting from the assumption of a separable form of the volatility function and in function of the state variables of the system.

3.1 Generalized dynamics

In the extended HJM framework, the dynamics of the instantaneous risk-free forward rate can be modeled through the following stochastic differential equation (SDE) for a fixed maturity T, under the extended T-forward measure:

$$\mathrm{d}f(t,T) = \mathbf{1}_{\{t\leq T\}} \sum_{\alpha=1}^{n} [\sigma^{(\alpha)}(t,T)]^{\intercal} \mathrm{d}W^{(\alpha)}(t) \,, \tag{12}$$

where $\sigma^{(\alpha)}(t,T)$ is an N-dimensional vector of adapted processes, $W^{(\alpha)}(t) \equiv W_t^{(\alpha)}$ is an N-dimensional standard Brownian motion, the superscript $^{\intercal}$ denotes the transposed matrix and the indicator function $\mathbf{1}_{\{t\leq T\}}$ is introduced to ensure that the process is defined, and is constant (i.e. its volatility is zero), for times larger than T. In particular, when $t > T$, $f(t,T)$ remains equal to the value it has reached at time T: $f(t,T) = f(T,T) = r(T)$. This way, while in the classic HJM framework, for each T, the volatility $\sigma(t,T)$, and hence the process $f(t,T)$, are defined only for $t \leq T$, here $f(t,T)$ is defined for all pairs (t,T).
Given the instantaneous forward rate, the price of an extended zero-coupon bond at time t with maturity T is expressed through Eq. (6), from which, using Ito's lemma and Fubini's theorem, we deduce the following dynamics:

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = \begin{cases} r(t)\mathrm{d}t - \sum_{\alpha=1}^{n} \left(\int_t^T \sigma^{(\alpha)}(t,u)\mathbf{1}_{\{t\leq u\}}\mathrm{d}u \right)^{\intercal} \mathrm{d}W_t^{(\alpha)} & \text{for } t \leq T \\ r(t)\mathrm{d}t = \mathrm{d}B(t,T)/B(t,T) & \text{for } t > T \,. \end{cases} \tag{13}$$

If we now change the numeraire from the extended T-forward measure to the risk-neutral measure formulation, i.e.

$$\mathrm{d}W_t^{(\alpha)} = \mathrm{d}\widetilde{W}_t^{(\alpha)} + \left(\int_t^T \sigma^{(\alpha)}(t,u)\mathrm{d}u \right) \mathrm{d}t \,, \tag{14}$$

where the tilde symbol stands for the risk-neutral case, Eq. (12) reads

$$\mathrm{d}f(t,T) = \mathbf{1}_{\{t\leq T\}} \sum_{\alpha=1}^{n} [\sigma^{(\alpha)}(t,T)]^{\intercal} \left[\mathrm{d}\widetilde{W}_t^{(\alpha)} + \left(\int_t^T \sigma^{(\alpha)}(t,u)\mathrm{d}u \right) \mathrm{d}t \right] , \tag{15}$$

while Eq. (13), for $t \leq T$, becomes

$$\frac{\mathrm{d}P(t,T)}{P(t,T)} = r(t)\mathrm{d}t - \sum_{\alpha=1}^{n} \left(\int_t^T \sigma^{(\alpha)}(t,u)\mathbf{1}_{\{t\leq u\}}\mathrm{d}u \right)^{\intercal} \left[\mathrm{d}\widetilde{W}_t^{(\alpha)} + \left(\int_t^T \sigma^{(\alpha)}(t,u)\mathrm{d}u \right) \mathrm{d}t \right] , \quad t \leq T \,. \tag{16}$$

3.2 Constraints on the volatility process and dynamics of state variables

In order to guarantee tractability and a Markovian specification of the model, we specify the volatility process via the following separable form:

$$\sigma^{(\alpha)}(t,u) \equiv \sum_{\beta=1}^{n} h_t^{(\alpha,\beta)} g^{(\beta)}(t,u) \,, \quad \alpha = 1, ..., n \,, \tag{17}$$

where

$$g^{(\alpha)}(t,u) \equiv \exp\left(-\int_t^u \lambda^{(\alpha)}(y)\mathrm{d}y\right), \quad \alpha = 1,...,n\,. \tag{18}$$

Here, h is a matrix adapted process and λ is a deterministic array function.
Furthermore, we define the Ito stochastic vector process

$$X_t^{(\alpha)} \equiv \sum_{\beta=1}^{n}\int_0^t g^{(\beta)}(s,t)\left[(h_s^{\intercal})^{(\alpha,\beta)}\mathrm{d}\widetilde{W}_s^{(\beta)} + (h_s^{\intercal})^{(\alpha,\beta)}(h_s)^{(\alpha,\beta)}\left(\int_s^t g^{(\beta)}(s,y)\mathrm{d}y\right)\mathrm{d}s\right], \quad \alpha = 1,...,n\,, \tag{19}$$

the auxiliary matrix process

$$Y_t^{(\alpha,\beta)} \equiv \int_0^t g^{(\alpha)}(s,t)(h_s^{\intercal})^{(\alpha,\beta)}(h_s)^{(\alpha,\beta)}g^{(\beta)}(s,t)\mathrm{d}s\,, \quad \alpha,\beta = 1,...,n\,, \tag{20}$$

with $X_0^{(\alpha)} = Y_0^{(\alpha,\beta)} = 0\ \forall\alpha,\beta = 1,...,n$, and the vectorial deterministic function

$$G_0^{(\alpha)}(t,T_0,T_1) \equiv \int_{T_0}^{T_1} g^{(\alpha)}(t,y)\mathrm{d}y\,, \quad \alpha = 1,...,n\,. \tag{21}$$

The Markov processes $\{X_t^{(\alpha)}\}_{\alpha=1,...,n}$ and $\{Y_t^{(\alpha,\beta)}\}_{\alpha,\beta=1,...,n}$ are the state variables of the system, allowing a complete description of the instantaneous forward rates and discount factors. Their dynamics, under the risk-neutral measure, is specified by the following coupled SDEs:

$$\begin{cases} \mathrm{d}X_t^{(\alpha)} = \left(\sum_{\beta=1}^n Y_t^{(\alpha,\beta)} - \lambda^{(\alpha)}(t)X_t^{(\alpha)}\right)\mathrm{d}t + \sum_{\beta=1}^n h_t^{(\alpha,\beta)}\mathrm{d}\widetilde{W}_t^{(\beta)}\,, & \alpha = 1,...,n, \\ \mathrm{d}Y_t^{(\alpha,\beta)} = \left[(h_t^{\intercal}h_t)^{(\alpha,\beta)} - (\lambda^{(\alpha)}(t) + \lambda^{(\beta)}(t))Y_t^{(\alpha,\beta)}\right]\mathrm{d}t\,, & \alpha,\beta = 1,...,n, \end{cases} \tag{22}$$

where $(h_t^{\intercal}h_t)^{(\alpha,\beta)} \equiv \sum_{\mu=1}^n (h_t^{\intercal})^{(\alpha,\mu)}(h_t)^{(\mu,\beta)}$.

3.3 Reconstruction formulas of risk-free instantaneous forward rates and discount factors in risk-neutral measure formulation

Given the quantities defined in the above section, we can now express the dynamics of the (risk-free) instantaneous forward rate and the (risk-free) discount factors in the risk-neutral measure formulation. Regarding the instantaneous forward rate, integrating Eq. (15) yields

$$f(t,T) = f(0,T) + \sum_{\alpha=1}^n g^{(\alpha)}(t,T)\left[X_t^{(\alpha)} + \sum_{\beta=1}^n Y_t^{(\alpha,\beta)}G_0^{(\beta)}(t,t,T)\right], \quad t \le T\,, \tag{23}$$

with initial condition $f(0,T)$ given by the market, while, when $t > T$, given that $f(t,T) = r(T)$ we have

$$f(t,T) = f(0,T) + \sum_{\alpha=1}^n X_T^{(\alpha)}\,, \quad t > T\,. \tag{24}$$

With respect to the discount factors, by integrating Eq. (16) we get

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\left[-\sum_{\alpha=1}^n G_0^{(\alpha)}(t,t,T)\left(X_t^{(\alpha)} + \frac{1}{2}\sum_{\beta=1}^n Y_t^{(\alpha,\beta)}G_0^{(\beta)}(t,t,T)\right)\right], \quad t \le T\,. \tag{25}$$

4 A specific realization of the extended HJM model

Now we would like to implement a specific realization of the model, by choosing a suitable shape of the volatility function. In particular, we consider a square-root process of the form

$$h_t^{(\alpha,\beta)} = \sqrt{V_t}\sum_{\mu=1}^{n}(R^{\intercal})^{(\alpha,\mu)}\hat{\sigma}^{(\mu,\beta)} = \sqrt{V_t}\sum_{\mu=1}^{n}(R^{\intercal})^{(\alpha,\mu)}\delta^{(\mu,\beta)}\hat{\sigma}^{(\beta,\beta)} = \sqrt{V_t}R^{(\beta,\alpha)}\hat{\sigma}^{(\beta,\beta)}\ , \quad \alpha,\beta = 1,...,n. \tag{26}$$

Here, $\hat{\sigma}$ is a deterministic constant diagonal matrix,

$$\hat{\sigma} \equiv \begin{pmatrix} \sigma^{11} & 0 \\ 0 & \sigma^{22} \end{pmatrix}\ , \tag{27}$$

R is a lower triangular matrix such that $RR^{\intercal} = \rho$, where ρ is a (symmetric) correlation matrix,

$$R \equiv \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}\ , \tag{28}$$

and V_t is a deterministic function satisfying the following differential equation under the risk-neutral measure:

$$\mathrm{d}V_t = k(\theta - V_t)\mathrm{d}t\ , \quad V_0\ , \tag{29}$$

where k, θ and V_0 are three positive constants. Integrating Eq. (29) yields

$$V_t = \theta - (\theta - V_0)e^{-kt}\ . \tag{30}$$

Furthermore, we assume that $\{\lambda^{(\alpha)}\}_{\alpha=1,...n}$ in Eq. (18) is a deterministic constant, and hence the equation at issue reads

$$g^{(\alpha)}(t,u) = e^{-\lambda^{(\alpha)}(u-t)}\ , \quad \alpha = 1,...,n\ , \tag{31}$$

while Eq. (21), with $T_0 = (T-x)\vee t$ and $T_1 = T \vee t$, becomes

$$\begin{aligned} G_0^{(\alpha)}(t,(T-x)\vee t, T\vee t) &= \int_{(T-x)\vee t}^{T\vee t} g^{(\alpha)}(t,u)\mathrm{d}u = \int_{(T-x)\vee t}^{T\vee t} e^{-\lambda^{(\alpha)}(u-t)}\mathrm{d}u \\ &= \frac{1}{\lambda^{(\alpha)}}\left(e^{-\lambda^{(\alpha)}((T-x)\vee t-t)} - e^{-\lambda^{(\alpha)}(T\vee t-t)}\right) \\ &= \frac{\Lambda_F^{(\alpha)}(t,(T-x)\vee t, T\vee t)}{\hat{\sigma}^{(\alpha,\alpha)}}\ , \quad \alpha = 1,...,n\ , \end{aligned} \tag{32}$$

where we have defined

$$\Lambda_F^{(\alpha)}(t,(T-x)\vee t, T\vee t) \equiv \frac{\hat{\sigma}^{(\alpha,\alpha)}}{\lambda^{(\alpha)}}\left(e^{-\lambda^{(\alpha)}((T-x)\vee t-t)} - e^{-\lambda^{(\alpha)}(T\vee t-t)}\right)\ , \quad \alpha = 1,...,n\ . \tag{33}$$

In the subsequent sections, for simplicity of notation, we will drop the parentheses in the superscripts.

4.1 Black-like formulas for the valuation of RFR vanilla derivatives

The specific realization of the HJM model defined above allows us to draw expressions for the dynamics of the risk-free forward and swap rates, as well as pricing formulas for caps and European swaptions derived from them, which have been used for the model calibration. It is market practice to value caps and swaptions with a Black-like formula, and we will adopt the same procedure here. In particular, in the swap case, we will obtain approximated expressions based on the initial freezing technique [14]. Since, as we will see shortly, the dynamics of the RFR forward and swap rates are identical with respect to the corresponding measures (they are both martingales), the pricing of a cap and of a European swaption constitute essentially the same mathematical problem. The following sections will be dedicated to this.

4.1.1 Risk-free forward rate dynamics

Let us consider a risk-free forward rate with maturity T and tenor x, $R_t(T-x,T)$. Because of its own definition (9), this is a martingale under the corresponding extended T-forward measure and its Q^T-dynamics, which is defined for every time t, including $t > T$, can be written as follows:

$$\frac{\mathrm{d}R_t(T-x,T)}{R_t(T-x,T)+1/x} = \left(\int_{T-x}^{T} \sigma(t,u)\mathbf{1}_{\{t\leq u\}}\mathrm{d}u\right)^{\intercal}\mathrm{d}W_t$$
$$= \left(\int_{(T-x)\vee t}^{T\vee t} \sigma(t,u)\mathrm{d}u\right)^{\intercal}\mathrm{d}W_t\ , \tag{34}$$

which, by exploiting Eqs. (17) and (21), becomes

$$\frac{\mathrm{d}R_t(T-x,T)}{R_t(T-x,T)+1/x} = \sum_{\alpha,\beta=1}^{n}(h_t^{\intercal})^{\alpha,\beta}G_0^{\beta}(t,(T-x)\vee t, T\vee t)\mathrm{d}W_t^{\alpha}\ . \tag{35}$$

From this equation, substituting Eqs. (26) and (32), it follows that

$$\mathrm{d}R_t(T-x,T) = \left(R_t(T-x,T)+\frac{1}{x}\right)\sqrt{V_t}\sum_{\alpha,\beta=1}^{n}\hat{\sigma}^{\beta,\beta}R^{\alpha,\beta}G_0^{\beta}(t,(T-x)\vee t,T\vee t)\mathrm{d}W_t^{\alpha}$$
$$= \left(R_t(T-x,T)+\frac{1}{x}\right)\sqrt{V_t}\sum_{\alpha,\beta=1}^{n}\Lambda_F^{\beta}(t,(T-x)\vee t,T\vee t)R^{\alpha,\beta}\mathrm{d}W_t^{\alpha}\ . \tag{36}$$

4.1.2 Risk-free swap rate dynamics

Let us consider a swap where the floating leg pays, at times $\{T_{a+1},...,T_b\}$, a rate obtained by compounding the daily fixing of the RFR from T_{k-1} to T_k, and where the fixed leg pays, at times $\{T_{\bar{a}+1},...,T_{\bar{b}}\}$, a fixed rate K. We assume that $T_{\bar{a}} = T_a$ and $T_{\bar{b}} = T_b$. Then, approximating the floating-leg payment at each time T_k by $R(T_{k-1},T_k)$ times the corresponding year fraction τ_k, and denoting by $\bar{\tau}_{\bar{k}}$ the year fraction associated with the fixed-leg interval $[T_{\bar{k}-1},T_{\bar{k}})$, we can express the forward swap rate as

$$S_t^{a,b}(x,\bar{x}) = \frac{\sum_{k=a+1}^{b}\tau_k P(t,T_k)R_t(T_{k-1},T_k)}{\sum_{\bar{k}=\bar{a}+1}^{\bar{b}}\bar{\tau}_{\bar{k}}P(t,T_{\bar{k}})} = \sum_{k=a+1}^{b} w_{k,t}^{a,b}(x,\bar{x})R_t(T_{k-1},T_k)\ , \tag{37}$$

which represents the fixed rate K that nullifies the swap value at time t. Here,

$$w_{k,t}^{a,b}(x,\bar{x}) \equiv \frac{\tau_k P(t,T_k)}{\sum_{\bar{k}=\bar{a}+1}^{\bar{b}}\bar{\tau}_{\bar{k}}P(t,T_{\bar{k}})} \tag{38}$$

and we have set $x \equiv \tau_k$, $\bar{x} \equiv \bar{\tau}_{\bar{k}}$. It is due to note that, since the extended zero-coupon bond prices and the forward rates are defined for all values of t, the swap rate formula above is also

defined for all t. For $t > T_a$, similarly to the zero-coupon bond case, this formula represents the value of a self-financing investment strategy at time t where all cash flows are reinvested (or financed, in the case of negative cash flows) at the risk-free rate, to roll them forward to the present time. We also remark that, when $t \leq T_a$, the swap price remains the same if we switch from forward-looking to backward-looking rates.
Now, starting from Eq. (37) and adopting the initial freezing approximation technique, from which

$$w_{k,t}^{a,b}(x,\bar{x}) \simeq w_{k,0}^{a,b}(x,\bar{x}) \,, \tag{39}$$

under the forward swap measure $Q^{a,b}$ associated with its annuity numeraire[1] $A^{a,b}(t;\bar{x}) \equiv \sum_{k=\bar{a}+1}^{\bar{b}} \bar{\tau}_k P(t,T_k)$, we can write

$$\begin{aligned}
\mathrm{d}S_t^{a,b}(x,\bar{x}) &\simeq \sum_{k=a+1}^{b} w_{k,0}^{a,b}(x,\bar{x})\mathrm{d}R_t(T_{k-1},T_k) \\
&= \sum_{k=a+1}^{b} w_{k,0}^{a,b}\left(R_t(T_{k-1},T_k)+\frac{1}{\tau_k}\right)\sqrt{V_t}\sum_{\alpha,\beta=1}^{n}\Lambda_F^{\beta}(t,T_{k-1}\vee t,T_k\vee t)R^{\alpha,\beta}\mathrm{d}W_t^{\alpha} \\
&= \left(S_t^{a,b}(x,\bar{x})+\psi^{a,b}(x,\bar{x})\right)\sqrt{V_t}\sum_{k=a+1}^{b}\delta_{a,b}^{k}\sum_{\alpha,\beta=1}^{n}\Lambda_F^{\beta}(t,T_{k-1}\vee t,T_k\vee t)R^{\alpha,\beta}\mathrm{d}W_t^{\alpha} \,,
\end{aligned} \tag{40}$$

where we have made use of Eqs. (36) and (38) and we have adopted the following definitions:

$$\psi^{a,b}(x,\bar{x}) \equiv \frac{\sum_{j=a+1}^{b} P(0,T_j)}{\sum_{j=\bar{a}+1}^{\bar{b}} \bar{\tau}_j P(0,T_j)} \tag{41}$$

and

$$\delta_{a,b}^{j}(x) \equiv \frac{\tau_j P(0,T_j)\left(R_0(T_{j-1},T_j)+\frac{1}{\tau_j}\right)}{\sum_{k=a+1}^{b}\tau_k P(0,T_k)\left(R_0(T_{k-1},T_k)+\frac{1}{\tau_k}\right)} = \frac{P(0,T_{j-1})}{\sum_{k=a+1}^{b} P(0,T_{k-1})} \,, \tag{42}$$

being $R_0(T_{j-1},T_j) = (P(0,T_{j-1})/P(0,T_j)-1)/\tau_j$. From Eq. (40) the dynamics of the swap rate under the measure $Q^{a,b}$ can be written as follows:

$$\frac{\mathrm{d}S_t^{a,b}(x,\bar{x})}{S_t^{a,b}(x,\bar{x})+\psi^{a,b}(x,\bar{x})} \simeq \sqrt{V_t}\sum_{\alpha,\beta=1}^{n}\Lambda_{S^{a,b}}^{\beta}(t,T_a\vee t,T_b\vee t)R^{\alpha,\beta}\mathrm{d}W_t^{\alpha} \,, \tag{43}$$

where

$$\Lambda_{S^{a,b}}^{\beta}(t,T_a,T_b) \equiv \sum_{k=a+1}^{b}\delta_{a,b}^{k}\Lambda_F^{\beta}(t,T_{k-1},T_k) \,, \quad t \leq T_a \,. \tag{44}$$

4.1.3 Valuation of an RFR cap

For each application period $[T-x,T]$ of Sec. 4.1.1, we can define two distinct caplets with strike K and paying off at time T:

[1] Similarly to LIBOR-based swap rates, also the risk-free swap rates are martingales under the forward swap measure $Q^{a,b}$, and thus we can assume specific dynamics of the RFR swap rate under such measure and price swaptions accordingly. For example, by assuming a geometric Brownian motion, we can write the swaptions price through the usual Black formula.

1. a forward-looking caplet with payoff $[R(T-x)-K]^+$;
2. a backward-looking caplet with payoff $[R(T)-K]^+$.

The two caplets mainly differ in that the forward-looking one is known at the beginning of the application period, $T-x$, while the backward-looking one is known only at the end of the application period, T. In order to evaluate the two caplets, we start from Eq. (36), which represents the dynamics of the forward rate $R_t(T-x,T)$ under the extended T-forward measure, and we write:

$$\begin{aligned}
&\mathrm{d}\ln\left(R_t(T-x,T)+\frac{1}{x}\right)\mathrm{d}\ln\left(R_t(T-x,T)+\frac{1}{x}\right)\\
&= V_t\sum_{\alpha,\beta=1}^{n}\Lambda_F^{\beta}R^{\alpha,\beta}\mathrm{d}W_t^{\alpha}\sum_{\nu,\mu=1}^{n}\Lambda_F^{\mu}R^{\nu,\mu}\mathrm{d}W_t^{\nu} = V_t\sum_{\alpha,\beta,\nu,\mu=1}^{n}\Lambda_F^{\beta}\Lambda_F^{\mu}R^{\alpha,\beta}R^{\nu,\mu}\delta^{\alpha,\nu}\mathrm{d}t\\
&= V_t\sum_{\alpha,\beta,\mu=1}^{n}\Lambda_F^{\beta}\Lambda_F^{\mu}R^{\alpha,\beta}R^{\alpha,\mu}\mathrm{d}t = V_t\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}\rho^{\beta,\mu}\Lambda_F^{\mu}\mathrm{d}t\,,
\end{aligned}\tag{45}$$

where $\Lambda_F \equiv \Lambda_F(t,(T-x)\vee t, T\vee t)$ for simplicity of notation.
Now let us calculate the volatilies of the two caplets. For the forward-looking one, defined for $t\le T-x$, we have

$$v_{\mathrm{FL}}^2(t) = \int_t^{T-x} V_u\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}(u,T-x,T)\rho^{\beta,\mu}\Lambda_F^{\mu}(u,T-x,T)\mathrm{d}u \tag{46}$$

which, by substituting expressions (30) and (33) and performing the integral, reads

$$\begin{aligned}
v_{\mathrm{FL}}^2(t) = &\sum_{\beta,\mu=1}^{n}\frac{\hat\sigma^{\beta,\beta}\hat\sigma^{\mu,\mu}}{\lambda^{\beta}\lambda^{\mu}}\rho^{\beta,\mu}\left(e^{-\lambda^{\beta}(T-x)}-e^{-\lambda^{\beta}T}\right)\left(e^{-\lambda^{\mu}(T-x)}-e^{-\lambda^{\mu}T}\right)\\
&\times\left[\frac{\theta}{\lambda^{\beta}+\lambda^{\mu}}\left(e^{(\lambda^{\beta}+\lambda^{\mu})(T-x)}-e^{(\lambda^{\beta}+\lambda^{\mu})t}\right)-\frac{\theta-V_0}{\lambda^{\beta}+\lambda^{\mu}-K}\left(e^{(\lambda^{\beta}+\lambda^{\mu}-K)(T-x)}-e^{(\lambda^{\beta}+\lambda^{\mu}-K)t}\right)\right].
\end{aligned}\tag{47}$$

Analogously, the volatility of the backward-looking caplet, defined for $t\le T$, is written as

$$\begin{aligned}
v_{\mathrm{BL}}^2(t) &= \int_t^{T} V_u\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}(u,(T-x)\vee u,T\vee u)\rho^{\beta,\mu}\Lambda_F^{\mu}(u,(T-x)\vee u,T\vee u)\mathrm{d}u\\
&= \int_t^{T-x} V_u\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}(u,T-x,T)\rho^{\beta,\mu}\Lambda_F^{\mu}(u,T-x,T)\mathrm{d}u\\
&+ \int_{T-x}^{T} V_u\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}(u,u,T)\rho^{\beta,\mu}\Lambda_F^{\mu}(u,u,T)\mathrm{d}u\\
&= v_{\mathrm{FL}}^2(t) + \int_{T-x}^{T} V_u\sum_{\beta,\mu=1}^{n}\Lambda_F^{\beta}(u,u,T)\rho^{\beta,\mu}\Lambda_F^{\mu}(u,u,T)\mathrm{d}u\,,
\end{aligned}\tag{48}$$

where we have used expression (46). Exploiting again definitions (30) and (33) and solving the integral, Eq. (48) becomes

$$
\begin{aligned}
v_{\mathrm{BL}}^2(t) = v_{\mathrm{FL}}^2(t) + \sum_{\beta,\mu=1}^{n} \frac{\hat{\sigma}^{\beta,\beta}\hat{\sigma}^{\mu,\mu}}{\lambda^\beta\lambda^\mu}\rho^{\beta,\mu} \\
\times \Bigg\{ \theta \bigg[x - \frac{1}{\lambda^\beta}\left(1 - e^{-\lambda^\beta x}\right) - \frac{1}{\lambda^\mu}\left(1 - e^{-\lambda^\mu x}\right) + \frac{1}{\lambda^\beta + \lambda^\mu}\left(1 - e^{-(\lambda^\beta+\lambda^\mu)x}\right) \bigg] \\
+ (\theta - V_0)e^{-Kt} \bigg[\frac{1}{K}\left(1 - e^{-Kx}\right) + \frac{1}{\lambda^\beta - K}\left(1 - e^{-(\lambda^\beta - K)x}\right) + \frac{1}{\lambda^\mu - K}\left(1 - e^{-(\lambda^\mu - K)x}\right) \\
- \frac{1}{\lambda^\beta + \lambda^\mu - K}\left(1 - e^{-(\lambda^\beta+\lambda^\mu-K)x}\right) \bigg] \Bigg\} . \qquad (49)
\end{aligned}
$$

We are now ready to deduce the prices of both caplets, which are of a Black-like form given the dynamics of $R_t(T-x,T)$. If we introduce the Black formula [15]:

$$
\mathrm{Black}(F,K,v) \equiv F\Phi\left(\frac{\ln(F/K) + \frac{1}{2}v}{\sqrt{v}}\right) - K\Phi\left(\frac{\ln(F/K) - \frac{1}{2}v}{\sqrt{v}}\right) , \qquad (50)
$$

with Φ being the standard Gaussian cumulative distribution function, we obtain

$$
C_{\mathrm{FL}}(t,T-x,T) = P(t,T) \times \mathrm{Black}\left(R_t(T-x,T) + \frac{1}{x}, K + \frac{1}{x}, v_{\mathrm{FL}}^2(t)\right) , \quad t \le T - x , \qquad (51)
$$

and

$$
C_{\mathrm{BL}}(t,T-x,T) = P(t,T) \times \mathrm{Black}\left(R_t(T-x,T) + \frac{1}{x}, K + \frac{1}{x}, v_{\mathrm{BL}}^2(t)\right) , \quad t \le T , \qquad (52)
$$

for the forward-looking and the backward-looking RFR caplet, respectively. Before concluding, just note from Eq. (49) that $v_{\mathrm{BL}}^2(t) \ge v_{\mathrm{FL}}^2(t)$ for $t \le T - x$, from which it follows that $C_{\mathrm{BL}}(t,T-x,T) \ge C_{\mathrm{FL}}(t,T-x,T)$. This implies that the backward-looking caplet is always more expensive than the forward-looking one.

4.1.4 Valuation of an RFR swaption

A European (payer or receiver) RFR swaption is the option to enter a spot RFR swap on the swaption's maturity date. Adopting the same notations of Sec. 4.1.2, we denote by $\{T_{a+1},...,T_b\}$ the floating-leg dates and by $\{T_{\bar{a}+1},...,T_{\bar{b}}\}$ the fixed-leg ones, with $T_{\bar{a}} = T_a$ and $T_{\bar{b}} = T_b$. Moreover, K is the strike and T_a is the swaption's maturity. From Eq. (43) we have

$$
\begin{aligned}
&\mathrm{d}\ln\left(S_t^{a,b}(x,\bar{x}) + \psi^{a,b}(x,\bar{x})\right)\mathrm{d}\ln\left(S_t^{a,b}(x,\bar{x}) + \psi^{a,b}(x,\bar{x})\right) \\
&\simeq V_t \sum_{\alpha,\beta=1}^{n} \Lambda_{S^{a,b}}^{\beta} R^{\alpha,\beta}\mathrm{d}W_t^\alpha \sum_{\nu,\mu=1}^{n} \Lambda_{S^{a,b}}^{\mu} R^{\nu,\mu}\mathrm{d}W_t^\nu = V_t \sum_{\alpha,\beta,\nu,\mu=1}^{n} \Lambda_{S^{a,b}}^{\beta}\Lambda_{S^{a,b}}^{\mu} R^{\alpha,\beta}R^{\nu,\mu}\delta^{\alpha,\nu}\mathrm{d}t \\
&= V_t \sum_{\alpha,\beta,\nu,\mu=1}^{n} \Lambda_{S^{a,b}}^{\beta}\Lambda_{S^{a,b}}^{\mu} R^{\alpha,\beta}R^{\alpha,\mu}\mathrm{d}t = V_t \sum_{\beta,\mu=1}^{n} \Lambda_{S^{a,b}}^{\beta}\rho^{\beta,\mu}\Lambda_{S^{a,b}}^{\mu}\mathrm{d}t , \qquad (53)
\end{aligned}
$$

where $\Lambda_{S^{a,b}} \equiv \Lambda_{S^{a,b}}(t, T_a \vee t, T_b \vee t)$ for simplicity of notation. The swaption volatility is then computed as follows:

$$
v_{S^{a,b}}^2(t) = \int_t^{T_a} V_u \sum_{\beta,\mu=1}^{n} \Lambda_{S^{a,b}}^{\beta}(u,T_a,T_b)\rho^{\beta,\mu}\Lambda_{S^{a,b}}^{\mu}(u,T_a,T_b)\mathrm{d}u , \qquad (54)
$$

which, by substituting expressions (30), (33) and (44) and performing the integral, becomes

$$
\begin{aligned}
v^2_{S^{a,b}}(t) = & \sum_{k,j=a+1}^{b} \delta^k_{a,b}\delta^j_{a,b} \sum_{\beta,\mu=1}^{n} \frac{\hat{\sigma}^{\beta,\beta}\hat{\sigma}^{\mu,\mu}}{\lambda^\beta\lambda^\mu}\rho^{\beta,\mu}\left(e^{-\lambda^\beta T_{k-1}} - e^{-\lambda^\beta T_k}\right)\left(e^{-\lambda^\mu T_{j-1}} - e^{-\lambda^\mu T_j}\right) \\
& \times \left[\frac{\theta}{\lambda^\beta+\lambda^\mu}\left(e^{(\lambda^\beta+\lambda^\mu)T_a} - e^{(\lambda^\beta+\lambda^\mu)t}\right) - \frac{\theta - V_0}{\lambda^\beta+\lambda^\mu-K}\left(e^{(\lambda^\beta+\lambda^\mu-K)T_a} - e^{(\lambda^\beta+\lambda^\mu-K)t}\right)\right] \\
= & \sum_{\beta,\mu=1}^{n} \kappa^{\beta,\mu}(t,T_a) \sum_{k,j=a+1}^{b} \delta^k_{a,b}\delta^j_{a,b}\left(e^{-\lambda^\beta T_{k-1}} - e^{-\lambda^\beta T_k}\right)\left(e^{-\lambda^\mu T_{j-1}} - e^{-\lambda^\mu T_j}\right) \\
= & \sum_{\beta,\mu=1}^{n} \kappa^{\beta,\mu}(t,T_a)\Lambda^\beta_{S^{a,b}}(0,T_a,T_b)\Lambda^\mu_{S^{a,b}}(0,T_a,T_b)\ , \qquad (55)
\end{aligned}
$$

where we have exploited the following definition:

$$
\begin{aligned}
\kappa^{\beta,\mu}(t,T_a) \equiv \rho^{\beta,\mu} \times & \left[\frac{\theta}{\lambda^\beta+\lambda^\mu}\left(e^{(\lambda^\beta+\lambda^\mu)T_a} - e^{(\lambda^\beta+\lambda^\mu)t}\right)\right. \\
& \left. - \frac{\theta - V_0}{\lambda^\beta+\lambda^\mu-K}\left(e^{(\lambda^\beta+\lambda^\mu-K)T_a} - e^{(\lambda^\beta+\lambda^\mu-K)t}\right)\right] . \qquad (56)
\end{aligned}
$$

Given the volatility just computed above, we can now write down the price of a European RFR swaption through a Black-like formula of the type:

$$
\Pi^{a,b}_t = A^{a,b}(t;\bar{x}) \times \mathrm{Black}(S^{a,b}_t(x,\bar{x}) + \psi^{a,b}(x,\bar{x}), K + \psi^{a,b}(x,\bar{x}), v^2_{S^{a,b}}(t))\ , \qquad (57)
$$

where $A^{a,b}(t;\bar{x})$ is the annuity and we have used Eq. (50). We would like to notice that, contrary to the cap case, the valuation formulas for swaps and swaptions present no difference between the forward-looking and the backward-looking situation.

4.2 Specific realization of Markov processes dynamics

We conclude this part by writing a specific formulation of the risk-neutral dynamics of the involved Markov processes (22), given the assumption (26):

$$
\begin{cases}
\mathrm{d}X^{(\alpha)}_t = \left(\sum_{\beta=1}^{n} Y^{\alpha,\beta}_t - \lambda^\alpha(t)X^\alpha_t\right)\mathrm{d}t + \sqrt{V_t}\sum_{\beta=1}^{n} R^{\beta,\alpha}\hat{\sigma}^{\beta,\beta}\mathrm{d}\widetilde{W}^\beta_t\ , & \alpha = 1, ..., n, \\
\mathrm{d}Y^{(\alpha,\beta)}_t = \left[V_t\hat{\sigma}^{\alpha,\alpha}\hat{\sigma}^{\beta,\beta}\rho^{\alpha,\beta} - \left(\lambda^\alpha(t) + \lambda^\beta(t)\right)Y^{\alpha,\beta}_t\right]\mathrm{d}t\ , & \alpha,\beta = 1, ..., n.
\end{cases} \qquad (58)
$$

5 Model calibration and numerical examples

In this section, we will present some numerical results obtained by adopting the HJM model described in the previous section. In particular, we will make use of the formula (57) for the RFR swaptions price in order to calibrate the model to the at-the-money swaptions volatility matrix. Next, we will use the calibrated model to price a set of European and Bermuda swaptions by means of the Monte Carlo technique and we will compare these results, obtained from the HJM model, with the ones coming from the Hull-White model [16].

5.1 Market data and calibration

We have extracted the market data needed for our calculations from the Reuters platform on 25$^{\text{th}}$ May 2022. The discount and indexing curve is the OIS ESTER curve shown in Table 1.

Date	Rate	Discount	Date	Rate	Discount
5/26/2022	-0.5934%	1.0000163	8/27/2027	1.1312%	0.9425749
5/27/2022	-0.5929%	1.0000326	11/29/2027	1.1490%	0.9389370
6/3/2022	-0.5906%	1.0001461	2/28/2028	1.1656%	0.9353796
6/27/2022	-0.5836%	1.0005293	5/29/2028	1.1828%	0.9317582
7/27/2022	-0.5785%	1.0010019	8/28/2028	1.1999%	0.9280525
8/29/2022	-0.4612%	1.0012165	11/27/2028	1.2170%	0.9242756
11/28/2022	-0.2009%	1.0010310	2/27/2029	1.2342%	0.9204065
2/27/2023	0.0257%	0.9998045	5/28/2029	1.2505%	0.9165933
5/29/2023	0.2333%	0.9976473	8/27/2029	1.2663%	0.9127211
8/28/2023	0.3965%	0.9950250	11/27/2029	1.2821%	0.9087625
11/27/2023	0.5297%	0.9920559	2/27/2030	1.2984%	0.9047230
2/27/2024	0.6317%	0.9889678	5/27/2030	1.3149%	0.9007032
5/27/2024	0.7108%	0.9858952	8/27/2030	1.3331%	0.8964098
8/27/2024	0.7712%	0.9828069	11/27/2030	1.3520%	0.8920071
11/27/2024	0.8181%	0.9797596	2/27/2031	1.3710%	0.8875393
2/27/2025	0.8574%	0.9766975	5/27/2031	1.3888%	0.8831971
5/27/2025	0.8918%	0.9736682	8/27/2031	1.4064%	0.8787204
8/27/2025	0.9258%	0.9704260	11/27/2031	1.4233%	0.8742473
11/27/2025	0.9584%	0.9670788	2/27/2032	1.4397%	0.8697588
2/27/2026	0.9893%	0.9636472	5/27/2032	1.4561%	0.8653350
5/27/2026	1.0174%	0.9602660	5/29/2034	1.5766%	0.8287117
8/27/2026	1.0445%	0.9567267	5/27/2037	1.6806%	0.7787388
11/27/2026	1.0695%	0.9531600	5/27/2042	1.6911%	0.7149919
2/26/2027	1.0921%	0.9496219	5/27/2047	1.6187%	0.6692931
5/27/2027	1.1125%	0.9461293	5/27/2052	1.5469%	0.6309106

Table 1: OIS ESTER curve obtained from Reuters platform on 25$^{\text{th}}$ May 2022.

The volatility matrix has been also obtained from the Reuters platform on the same day, and refers to normal at-the-money swaptions volatilities for the EUR currency expressed in basis points. Such matrix is shown in Table 2.

The HJM model is calibrated to the matrix volatility using Eq. (57) and the Levenberg-Marquardt optimization algorithm [17], [18]. The resulting calibrated model parameters are listed in Table 3.

Start/Tenor	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	15Y	20Y	25Y	30Y
1Y	99.13	103.52	102.07	98.05	95.44	94.52	93.29	91.91	90.59	89.46	88.92	87.44	86.17	85.32
2Y	98.47	99.82	97.33	95.05	93.65	92.68	91.26	90.08	88.88	87.77	85.37	84.24	82.28	80.25
3Y	97.29	97.44	94.63	92.87	91.30	90.55	89.43	88.32	87.24	85.99	82.51	80.41	78.19	75.97
4Y	94.70	94.06	92.89	90.64	89.21	88.23	87.19	86.19	85.02	83.86	79.73	77.54	75.23	73.07
5Y	90.95	90.09	88.81	87.19	85.98	85.22	84.01	83.34	82.29	80.87	76.38	74.16	72.07	69.84
7Y	83.78	83.05	81.59	80.38	79.24	78.37	77.46	76.44	75.63	74.71	70.26	68.11	66.22	64.60
10Y	76.08	75.95	74.77	73.70	72.52	71.77	70.72	69.60	68.64	67.75	63.87	61.17	59.39	58.01
15Y	67.53	67.16	66.66	66.13	65.33	64.47	63.48	62.36	61.59	60.77	56.14	53.69	52.41	50.92
20Y	61.15	60.93	60.79	60.53	60.33	59.54	58.75	57.84	57.01	56.11	51.93	49.07	47.66	45.91
25Y	58.39	58.01	57.87	57.43	57.10	56.24	55.09	54.01	53.14	52.46	48.09	44.94	43.12	41.34
30Y	56.19	56.08	55.88	55.42	54.87	53.94	52.72	51.59	50.32	49.72	44.66	41.67	39.75	37.89

Table 2: At-the-money swaptions normal volatilities (bp) calculated on Reuters platform on 25th May 2022.

Model parameters	Values
λ^1	0.0213891996660037
λ^2	0.0213891995259624
σ^1	0.0647222725706892
σ^2	0.0647222724725727
ρ	-0.3467774897613610
k	0.2467270593652540
θ	0.0074530089063929
V_0	0.0228443873170564

Table 3: HJM model parameters after the calibration.

5.2 European and Bermuda swaptions pricing results

In the following tables, the outcomes of the calculation of European (Table 4) and Bermuda (Table 5) swaptions prices are shown. In order to obtain these results, we have calibrated the Hull-White model to the market data presented in Tables 1 and 2, and we have used trinomial tree techniques for the pricing of both types of swaptions (see [19] if you are looking for more details on these techniques). Regarding the HJM model, instead, we have exploited the Monte Carlo pricing method. In particular, for Bermuda swaptions, the price has been obtained through the use of the Longstaff-Schwartz algorithm [20] for the valuation of path-dependent options. All calculations have been made by setting 25th May 2022 as the trade date.

Moreover, Figures 1 and 2 show the errors (both absolute and percentage in the first case, absolute in the second case) in the calculation of swaptions prices, in function of start and tenor, obtained through the HJM-FMM and the HW model, respectively, compared to the Black formula evaluations. Similarly, Figure 4 displays the absolute basis point and percentage errors of Bermuda swaptions prices between the HJM-FMM and the HW model.

Note from Figure 1 that the HJM-FMM pricing results for European swaptions are consistent with the Black evaluations, as the percentage errors between the two models are within 5% for almost all the options analysed (lower panel). Moreover, as you can see from Table 4, the HJM pricing results are comparable to the Hull-White benchmark prices.

The greatest discrepancies occur for Bermuda swaptions (left part of Table 5), and this can be justified by the fact that we have used two different numerical procedures (trinomial tree and Monte Carlo with Longstaff-Schwartz algorithm) for the purpose of pricing a complex payoff, as a Bermuda swaption is. In fact, as tenor rises, the Bermuda swaptions HJM-FMM prices tend to increase significantly (Figure 3) and the absolute errors compared to HW grow quite accordingly, especially for long starts (upper panel of Figure 4). However, notice how the percentage errors, apart from the 1Y/1Y case, remain below 5% for all options (lower panel of Figure 4). Furthermore, the prices we have obtained for Bermuda swaptions are always greater than those of the corresponding European swaptions (right part of Table 5), as it should be.

We consider all of these observations as valid proofs of the goodness of our results.

Start	Tenor	Black Price	HW Price	HJM-FMM Price
1Y	1Y	0.389854%	0.348255%	0.408748%
1Y	5Y	1.826548%	1.637326%	1.899327%
1Y	10Y	3.287564%	3.003120%	3.450958%
1Y	15Y	4.679284%	4.122289%	4.707684%
1Y	20Y	5.870398%	5.075819%	5.771896%
1Y	30Y	7.969204%	6.632324%	7.489499%
2Y	1Y	0.540891%	0.517835%	0.554324%
2Y	5Y	2.498203%	2.429791%	2.567970%
2Y	10Y	4.483927%	4.433610%	4.655280%
2Y	15Y	6.242288%	6.078889%	6.360134%
2Y	20Y	7.862811%	7.477168%	7.804938%
2Y	30Y	10.433598%	9.721322%	10.124658%
5Y	1Y	0.760347%	0.755671%	0.744168%
5Y	5Y	3.450557%	3.479480%	3.437479%
5Y	10Y	6.176034%	6.289104%	6.212237%
5Y	15Y	8.354951%	8.606813%	8.516025%
5Y	20Y	10.386176%	10.572680%	10.490864%
5Y	30Y	13.692945%	13.620498%	13.614382%
10Y	1Y	0.813138%	0.818935%	0.811179%
10Y	5Y	3.717014%	3.754015%	3.734509%
10Y	10Y	6.636392%	6.791823%	6.808297%
10Y	15Y	9.026485%	9.307336%	9.412811%
10Y	20Y	11.138757%	11.393784%	11.631489%
10Y	30Y	14.896690%	14.506314%	15.082173%
20Y	1Y	0.769952%	0.784578%	0.764951%
20Y	5Y	3.699699%	3.649005%	3.545769%
20Y	10Y	6.675343%	6.737284%	6.533710%
20Y	15Y	9.001449%	9.313294%	9.044026%
20Y	20Y	11.020587%	11.439752%	11.149528%
20Y	30Y	14.615141%	14.602037%	14.390096%
30Y	1Y	0.765745%	0.762565%	0.708423%
30Y	5Y	3.652050%	3.549917%	3.297440%
30Y	10Y	6.430524%	6.475036%	6.073636%
30Y	15Y	8.418892%	8.873113%	8.405603%
30Y	20Y	10.178421%	10.921943%	10.360204%
30Y	30Y	13.121632%	13.971738%	13.369577%

Table 4: European swaptions prices evaluated with Black formula, trinomial tree in Hull-White model and Monte Carlo in HJM-FMM model (*third, fourth and fifth column*, respectively).

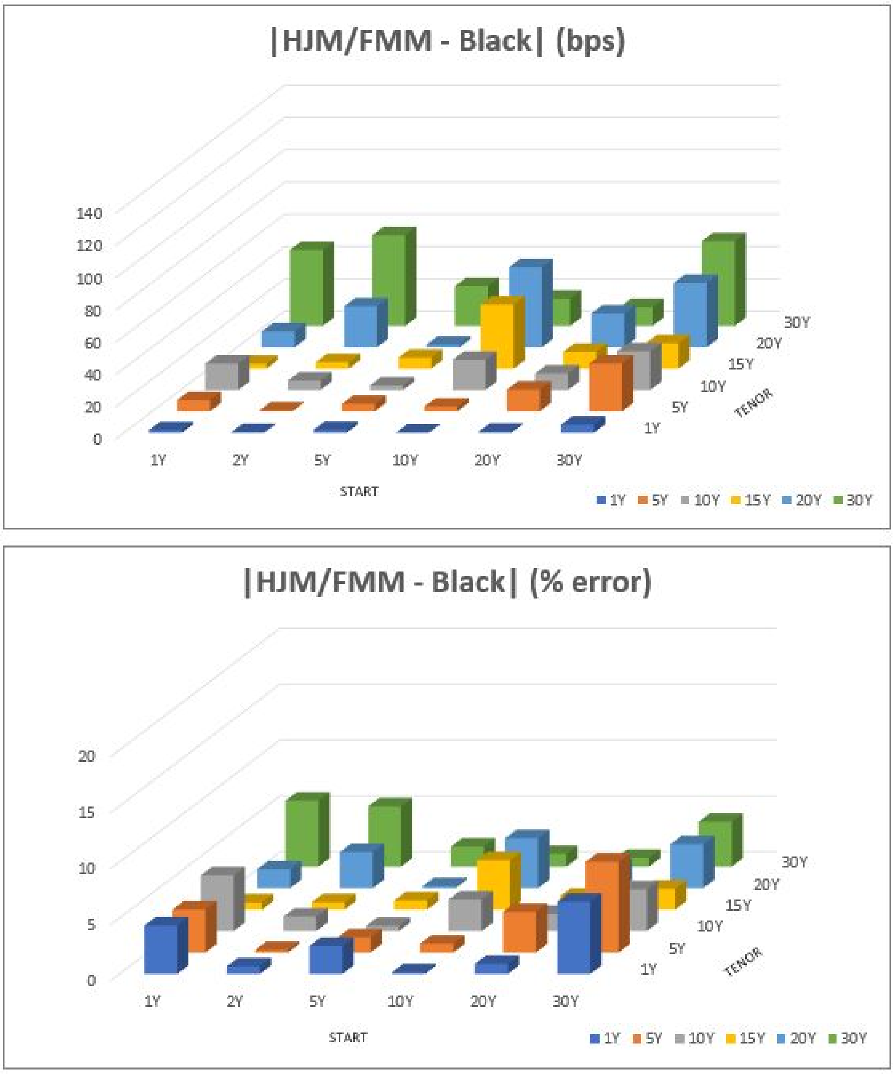

Figure 1: Absolute basis point difference (*upper*) and percentage error (*lower*) of European swaptions prices between the HJM-FMM model and the evaluation through the Black formula

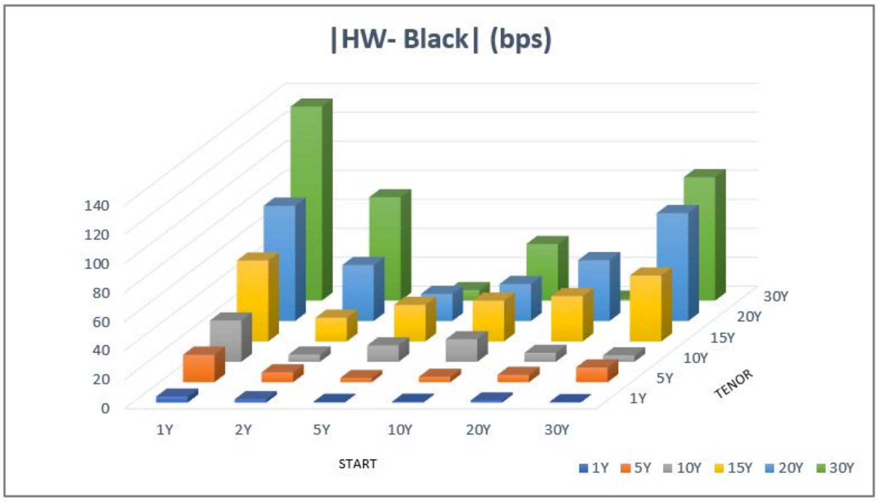

Figure 2: Absolute basis point errors of European swaptions prices between the HW model and the evaluation through Black formula.

Start	Tenor	Bermuda HW Price	Bermuda HJM-FMM Price	European HJM-FMM Price	Bermuda HJM-FMM Price
1Y	1Y	0.348255%	0.408748%	0.408748%	0.408748%
1Y	5Y	2.825827%	2.751778%	1.899327%	2.751778%
1Y	10Y	6.491251%	6.523418%	3.450958%	6.523418%
1Y	15Y	10.095990%	10.104821%	4.707684%	10.104821%
1Y	20Y	13.185259%	13.248416%	5.771896%	13.248416%
1Y	30Y	18.820955%	18.758287%	7.489499%	18.758287%
2Y	1Y	0.517835%	0.554324%	0.554324%	0.554324%
2Y	5Y	3.248694%	3.192102%	2.567970%	3.192102%
2Y	10Y	6.984831%	6.998196%	4.655280%	6.998196%
2Y	15Y	10.430235%	10.465146%	6.360134%	10.465146%
2Y	20Y	13.489772%	13.525484%	7.804938%	13.525484%
2Y	30Y	19.156863%	19.090555%	10.124658%	19.090555%
5Y	1Y	0.755671%	0.744168%	0.744168%	0.744168%
5Y	5Y	3.873051%	3.833295%	3.437479%	3.833295%
5Y	10Y	7.597204%	7.583686%	6.212237%	7.583686%
5Y	15Y	10.885533%	10.944404%	8.516025%	10.944404%
5Y	20Y	13.951111%	13.955817%	10.490864%	13.955817%
5Y	30Y	19.700769%	19.572498%	13.614382%	19.572498%
10Y	1Y	0.818935%	0.811179%	0.811179%	0.811179%
10Y	5Y	3.927197%	3.923193%	3.734509%	3.923193%
10Y	10Y	7.524099%	7.462343%	6.808297%	7.462343%
10Y	15Y	10.842598%	10.720053%	9.412811%	10.720053%
10Y	20Y	13.995898%	13.831620%	11.631489%	13.831620%
10Y	30Y	19.847005%	19.443431%	15.082173%	19.443431%
20Y	1Y	0.784578%	0.764951%	0.764951%	0.764951%
20Y	5Y	3.845991%	3.684877%	3.545769%	3.684877%
20Y	10Y	7.412319%	7.084911%	6.533710%	7.084911%
20Y	15Y	10.758392%	10.261963%	9.044026%	10.261963%
20Y	20Y	13.931864%	13.248902%	11.149528%	13.248902%
20Y	30Y	19.736820%	18.673497%	14.390096%	18.673497%
30Y	1Y	0.762565%	0.708423%	0.708423%	0.708423%
30Y	5Y	3.668129%	3.420019%	3.297440%	3.420019%
30Y	10Y	7.085254%	6.598128%	6.073636%	6.598128%
30Y	15Y	10.276920%	9.570426%	8.405603%	9.570426%
30Y	20Y	13.250599%	12.350248%	10.360204%	12.350248%
30Y	30Y	18.584454%	17.356583%	13.369577%	17.356583%

Table 5: Comparison between Bermuda swaptions prices evaluated with trinomial tree in Hull-White model and Monte Carlo in HJM-FMM model (*left*), and Bermuda swaptions HJM-FMM prices compared to the corresponding European swaptions HJM-FMM prices (*right*).

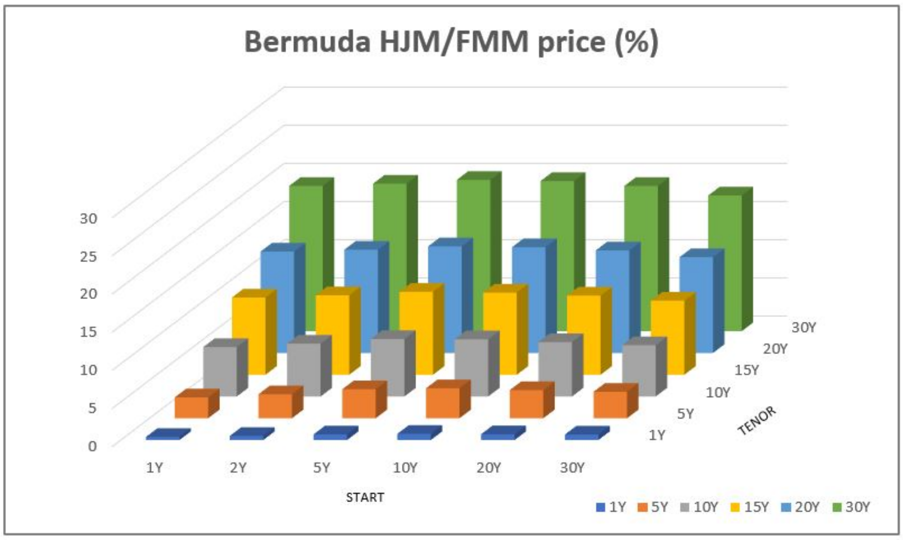

Figure 3: Percentage prices of Bermuda swaptions obtained with the HJM-FMM model.

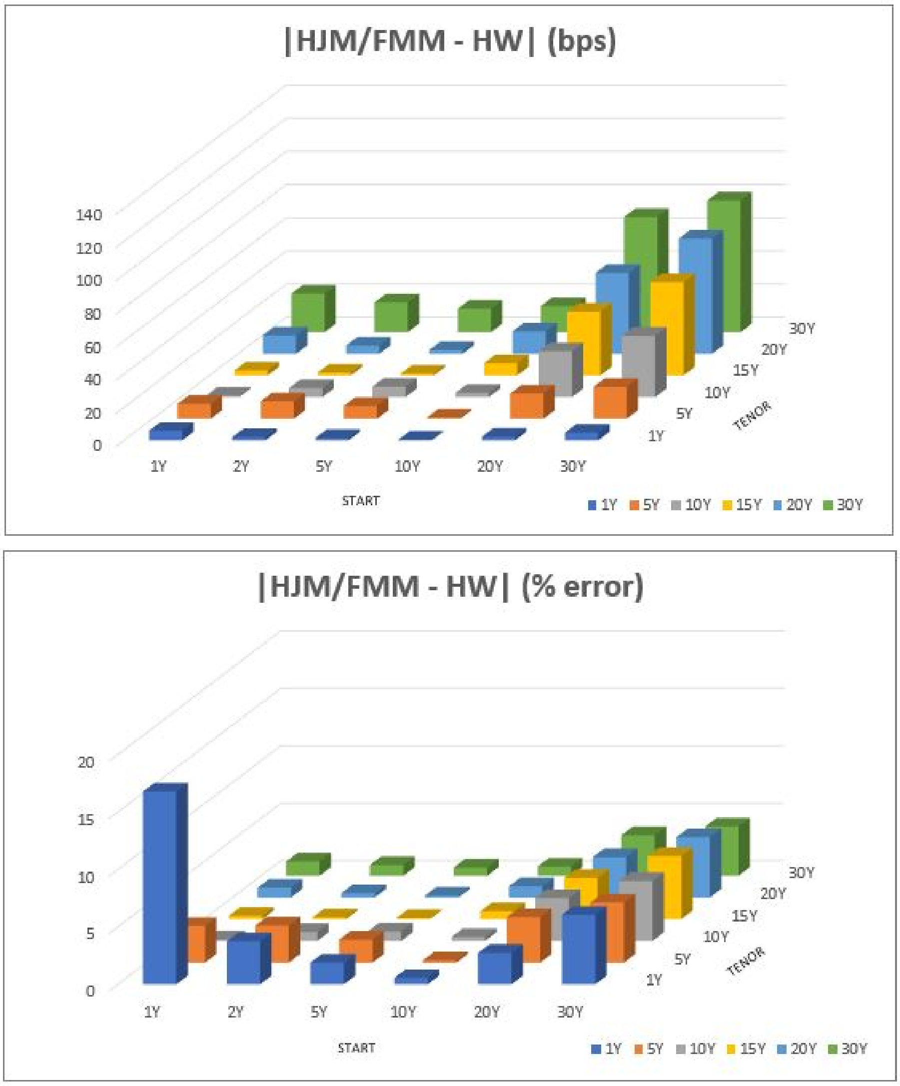

Figure 4: Absolute basis point difference (*upper*) and percentage error (*lower*) of Bermuda swaptions prices between the HJM-FMM model and the HW model.

6 Explicit swaptions price in HJM two-factor model

This last section is dedicated to the derivation of explicit formulas for the price of the European (payer and receiver) swaptions in the context of a HJM two-factor model. This has been basically done by extending the analysis performed in [4] and [5] for a HJM scalar model to a framework with two stochastic factors. Moreover, we will provide a specific formulation of these prices by adopting the assumptions made at the beginning of Sec. 4. Again, these valuation formulas have been used to calibrate our model.

6.1 Model and hypotheses

In a HJM two-factor model, the forward rate follows a vector dynamics, in that its evolution process relies on the presence of two Brownian motions instead of just one as in the scalar case. Let us assume that, for each one-dimensional standard Brownian motion $(W_t)_{0\leq t\leq T}$ involved in the process, a probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ exists with (augmented) filtration $\mathcal{F}_t$, and define $N_t \equiv \exp(\int_0^t r(s)\mathrm{d}s)$ as the cash-account numeraire forming a numeraire pair[2] with some measure $\mathbb{N}$. Then, the evolution of the instantaneous forward rate in the numeraire measure associated to N_t can be written as

$$\mathrm{d}f(t,u) = \sigma^{\intercal}(t,u)\nu(t,u)\mathrm{d}t + \sigma^{\intercal}(t,u)\mathrm{d}W_t \tag{59}$$

where we have defined, for simplicity,

$$\nu(t,u) \equiv \int_t^u \sigma(t,s)\mathrm{d}s\ , \tag{60}$$

with ν an increasing (vector) function of u, measurable and bounded. Moreover, we assume the following *separability hypothesis*:

H: *The function ν satisfies the condition $\nu(s,t_2)-\nu(s,t_1) = f(t_1,t_2)g(s)$, where g is a positive function.*

Note that, as σ and W_t are two-dimensional vectors, the product between them in Eq. (59) is intended to be a scalar product, and the same goes for the product between σ and ν.
It is worth saying that, when dealing with vector dynamics, we have to take into account the possible correlation between the different Brownian motions that take part in the process. In such a situation, it is common use to shift the correlation matrix from W_t to σ, in such a way that the motions can be treated as independent. Basically, if W_t^1 and W_t^2 are two correlated Brownian motions, and $\rho = CC^{\intercal}$ is the instantaneous correlation matrix, with C a 2×2 matrix and $C^{\intercal}$ its transpose, the tensor product between W_t^1 and W_t^2 is written as

$$\mathrm{d}W_t^1\mathrm{d}W_t^2 = \rho_{12}\mathrm{d}t = C_1C_2^{\intercal}\mathrm{d}t\ ,$$

where $\mathrm{d}t$ would have been the product between the two motions if they had been independent. However, in Eq. (59) we can treat the $\mathrm{d}W_t$ dynamics as uncorrelated, while we can insert the correlation by redefining the volatility σ as σC. With these assumptions, the instantaneous forward rate dynamics reads

$$\mathrm{d}f(t,u) = C^{\intercal}\sigma^{\intercal}(t,u)\bigg(\int_t^u \sigma(t,s)C\mathrm{d}s\bigg)\mathrm{d}t + C^{\intercal}\sigma^{\intercal}(t,u)\mathrm{d}W_t\ , \tag{61}$$

[2] See [22] for the definition of a numeraire pair.

where we have made definition (60) explicit.

In light of the above and in order to simplify the analysis, in the following treatment we will consider the Brownian motions as independent, while the correlation term will be implicitly included within the volatility factor.

6.2 The fundamental theorem and its proof

Before writing down the main result of our analysis, let us state two technical lemmas, which will be crucial for the following treatment.

Lemma 1. *The price of a zero-coupon bond in a HJM two-factor model can be written as*

$$P(t,u) = \frac{P(0,u)}{P(0,t)} \exp\Big(-\frac{1}{2}\int_0^t (\nu^2(s,u) - \nu^2(s,t))\mathrm{d}s - \int_0^t (\nu^\intercal(s,u) - \nu^\intercal(s,t))\mathrm{d}W_s \Big) . \quad (62)$$

Lemma 2. *In a HJM two-factor model, we have*

$$\exp\Big(-\int_0^t r(s)\mathrm{d}s \Big) = P(0,t)\exp\Big(-\int_0^t \nu^\intercal(s,t)\mathrm{d}W_s - \frac{1}{2}||\nu||^2 \Big) , \quad (63)$$

where $||\nu||^2 \equiv \int_0^t \nu^2(s,t)\mathrm{d}s$ *is the square norm of* ν.

For the proof of the two lemmas we refer the reader to [4]. Indeed, the proofs illustrated in the cited paper for the scalar situation are easily extendable to the two-factor case.

We are now ready to enunciate the following fundamental theorem.

Theorem 1. *Suppose we work in a HJM two-factor model with a volatility term of the form (H). Let* $0 \le t_0 \le t_1 \le \cdots \le t_n$ *and consider a swap which pays* $c_i > 0$ *at times* t_i $(1 \le i \le n$, $i \ne m, m+1)$. *The price of a European receiver swaption, with expiry* t_m *on the swap with forward payment of* $-c_{m+1} > 0$ *at time* t_{m+1}*, is given at time 0 by*

$$\sum_{i=m+1}^{n} c_i P(0,t_i) N\left(\frac{\kappa_1(-\alpha_{i2}) + \alpha_{i1}}{\sqrt{1 + [\kappa'_{12}(-\alpha_{i2})]^2}} \right) , \quad (64)$$

where N *is the cumulative distribution function of a standard normal distribution,* κ_1 *is the unique solution of*

$$\sum_{i=m+1}^{n} c_i P(0,t_i) \exp\Big(-\frac{1}{2}\sum_{k=1}^{2}\alpha_{ik}^2 - \alpha_{i1}\kappa_1 - \alpha_{i2}y_2 \Big) = 0 , \quad (65)$$

κ'_{12} *is the derivative of* κ_1 *with respect to* y_2,

$$\kappa'_{12}(x) \equiv \frac{\mathrm{d}\kappa_1}{\mathrm{d}y_2}(x) = -\frac{\sum_{k=m+1}^{n} \alpha_{k2} c_k P(0,t_k) \exp\big(-\frac{1}{2}||\alpha_k||^2 - \alpha_{k1}\kappa_1(x) - \alpha_{k2}x \big)}{\sum_{k=m+1}^{n} \alpha_{k1} c_k P(0,t_k) \exp\big(-\frac{1}{2}||\alpha_k||^2 - \alpha_{k1}\kappa_1(x) - \alpha_{k2}x \big)} \quad (66)$$

and the α_{ik} *are such that*

$$||\alpha_i||^2 \equiv \alpha_i^2 = \int_0^{t_m} (\nu(s,t_i) - \nu(s,t_m))^2 \mathrm{d}s , \quad (67)$$

with $||\alpha_i||^2 = \sum_{k=1}^{2} \alpha_{ik}^2$ *being the square norm of* α_i. *The price of a European payer swaption is*

$$-\sum_{i=m+1}^{n} c_i P(0,t_i) N\left(-\frac{\kappa_1(-\alpha_{i2}) + \alpha_{i1}}{\sqrt{1 + [\kappa'_{12}(-\alpha_{i2})]^2}} \right) . \quad (68)$$

Substantially, by writing the price of a zero-coupon bond in terms of a normally distributed random variable, the swaptions price is obtained as the sum of the discounted cash-flows multiplied by a Gaussian cumulative distribution function, with a term for each coupon plus one for the strike price. The pricing formula contains a parameter, κ_1, which is computed by solving a two-dimensional equation with as many exponential terms as the number of coupons plus one.

6.2.1 The proof

Let define the probability $\mathbb{P}^{\#}$ of density, with respect to $\mathbb{N}$,

$$\left.\frac{\mathrm{d}\mathbb{P}^{\#}}{\mathrm{d}\mathbb{N}}\right|_{\mathcal{F}_{t_m}} = \exp\left(-\int_0^{t_m} \nu^{\intercal}(s,t_m)\mathrm{d}W_s - \frac{1}{2}||\nu||^2 \right) \tag{69}$$

where $||\nu||^2 \equiv \int_0^{t_m} \nu^2(s,t_m)\mathrm{d}s$ is, again, the square norm of ν. By Girsanov's theorem [21], the process

$$W_t^{\#} = W_t + \int_0^t \nu(s,t_m)\mathrm{d}s$$

is a standard Brownian motion under $\mathbb{P}^{\#}$. Then, by Lemma 1, we have

$$\begin{aligned} P(t_m,t_i) &= \frac{P(0,t_i)}{P(0,t_m)} \exp\left(-\frac{1}{2}\alpha_i^2 - \int_0^{t_m} (\nu^{\intercal}(s,t_i) - \nu^{\intercal}(s,t_m))\mathrm{d}W_s^{\#} \right) \\ &= \frac{P(0,t_i)}{P(0,t_m)} \exp\left(-\frac{1}{2}\alpha_i^2 - \sum_{k=1}^{2} \alpha_{ik} X^k \right) \end{aligned}$$

where we have exploited the relation

$$\int_0^{t_m} (\nu^2(s,t_i) - \nu^2(s,t_m))\mathrm{d}s = \int_0^{t_m} (\nu(s,t_i) - \nu(s,t_m))^2\mathrm{d}s + 2\int_0^{t_m} \nu^{\intercal}(s,t_m)(\nu(s,t_i) - \nu(s,t_m))\mathrm{d}s$$

and X^k is a normally distributed stochastic variable with respect to $\mathbb{P}^{\#}$. Furthermore, condition (H) ensures that X is the same for all i.
Hence, the expression for the price of a receiver swaption can be written as

$$\begin{aligned} & E_{\mathbb{N}}\left[e^{-\int_0^{t_m} r(s)\mathrm{d}s} \left(\sum_{i=m+1}^{n} c_i P(t_m,t_i) \right)^+ \right] \\ &= E^{\#}\left[P(0,t_m) \left(\sum_{i=m+1}^{n} c_i \frac{P(0,t_i)}{P(0,t_m)} \exp\left(-\frac{1}{2}\alpha_i^2 - \sum_{k=1}^{2} \alpha_{ik} X^k \right) \right)^+ \right] \\ &= E^{\#}\left[\left(\sum_{i=m+1}^{n} c_i P(0,t_i) \exp\left(-\frac{1}{2}\alpha_i^2 - \sum_{k=1}^{2} \alpha_{ik} X^k \right) \right)^+ \right], \end{aligned} \tag{70}$$

where we have made use of Lemma 2 and Eq. (69). Moreover, $E_{\mathbb{N}}$ and $E^{\#}$ denote the expectation values with respect to probabilities $\mathbb{N}$ and $\mathbb{P}^{\#}$, respectively. By using the property that the expectation value of a distribution with multiple uncorrelated stochastic terms equals the product of the expectation values of the single terms, Eq. (70) reads

$$\frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\sum_{i=m+1}^{n} c_i P(0,t_i) \exp\left(-\frac{1}{2}\sum_{k=1}^{2} \alpha_{ik}^2 - \sum_{k=1}^{2} \alpha_{ik} y_k \right) \right)^+ e^{-\frac{1}{2}y_1^2} e^{-\frac{1}{2}y_2^2} \mathrm{d}y_1 \mathrm{d}y_2 \,. \tag{71}$$

Now we define

$$h(y_1, y_2) = \sum_{i=m+1}^{n} c_i P(0, t_i) \exp\left(-\frac{1}{2}\sum_{k=1}^{2} \alpha_{ik}^2 - \sum_{k=1}^{2} \alpha_{ik} y_k \right)$$

and we show that $h(y_1, y_2)$ is positive for $y_1 < \kappa_1$, where κ_1 is a function[3] of y_2: $\kappa_1 = \kappa_1(y_2)$. If we set

$$A_i = c_i P(0, t_i) \exp\left(-\frac{1}{2}\sum_{k=1}^{2} \alpha_{ik}^2 \right)$$

and

$$q_1(y_1, y_2) = \sum_{i=m+2}^{n} A_i \exp\left(-\sum_{k=1}^{2} (\alpha_{ik} - \alpha_{m+1,k}) y_k \right) + A_{m+1} \,,$$

$$q_2(y_1, y_2) = -\sum_{i=m+2}^{n} \sum_{k=1}^{2} \alpha_{ik} A_i \exp\left(-\sum_{k=1}^{2} (\alpha_{ik} - \alpha_{m+1,k}) y_k \right) - \sum_{k=1}^{2} \alpha_{m+1,k} A_{m+1} \,,$$

then

$$h(y_1, y_2) = \exp\left(-\sum_{k=1}^{2} \alpha_{m+1,k} y_k \right) q_1(y_1, y_2)$$

and

$$h'(y_1, y_2) = \exp\left(-\sum_{k=1}^{2} \alpha_{m+1,k} y_k \right) q_2(y_1, y_2) \,.$$

By repeating an analogous procedure to the one in the proof of Theorem 3.1 in [4], we can prove that there exists κ_1 such that $h(\kappa_1, y_2) = 0$, with $h(y_1, y_2) > 0$ for $y_1 < \kappa_1$ and negative elsewhere. As a consequence, Eq. (71) can be rewritten as

$$\begin{aligned}
&\frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\kappa_1} \int_{-\infty}^{+\infty} \sum_{i=m+1}^{n} c_i P(0, t_i) \exp\left(-\frac{1}{2}\sum_{k=1}^{2} \alpha_{ik}^2 - \sum_{k=1}^{2} \alpha_{ik} y_k \right) e^{-\frac{1}{2}y_1^2} e^{-\frac{1}{2}y_2^2} \mathrm{d}y_1 \mathrm{d}y_2 \\
&= \sum_{i=m+1}^{n} c_i P(0, t_i) \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\kappa_1} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\alpha_{i1}^2 - \alpha_{i1} y_1 \right) e^{-\frac{1}{2}y_1^2} \\
&\times \exp\left(-\frac{1}{2}\alpha_{i2}^2 - \alpha_{i2} y_2 \right) e^{-\frac{1}{2}y_2^2} \mathrm{d}y_1 \mathrm{d}y_2 \\
&= \sum_{i=m+1}^{n} c_i P(0, t_i) \int_{-\infty}^{+\infty} N(\kappa_1(y_2) + \alpha_{i1}) g(y_2 + \alpha_{i2}) \mathrm{d}y_2 \,,
\end{aligned} \tag{72}$$

where N is again the normal cumulative distribution function and g is the Gaussian distribution. Due to the fact that we don't know how $\kappa_1(y_2)$ is made, Eq. (72) cannot be solved exactly, but only by making some approximations on this function. First of all, by Dini's theorem we know that $h(\kappa_1, y_2) = 0$ defines an implicit function $\kappa_1(y_2)$ in a neighborhood of y_2^0. We also note that, given the form of the Gaussian functions in the integrals, the relevant part of Eq. (72) is the one centered in $-\alpha_{i2}$, for each term of the summation. Then, we can approximate $\kappa_1(y_2)$ through a first-order Taylor expansion around $-\alpha_{i2}$. In this way, we will have as many κ_1 as are the α_{i2}, that is, as are the swap payment dates. In formulas:

$$\kappa_1(y_2) = \kappa_1^i(y_2) \simeq \kappa_1(-\alpha_{i2}) + \frac{\mathrm{d}\kappa_1}{\mathrm{d}y_2}(-\alpha_{i2})(y_2 + \alpha_{i2}) \,,$$

[3] Here we choose to vary y_1 while maintaining y_2 fixed. However, such choice is completely arbitrary, as we could potentially decide to do the opposite, i.e. vary y_2 and leave y_1 fixed.

which can be written explicitly since the derivatives of an implicit function are known.
With these approximations, the integral in Eq. (72) can be solved analytically, for each term of the summation. The final result for the price of a European receiver swaption in the two-dimensional case reads

$$\sum_{i=m+1}^{n} c_i P(0,t_i) N\left(\frac{\kappa_1(-\alpha_{i2}) + \alpha_{i1}}{\sqrt{1+[\kappa'_{12}(-\alpha_{i2})]^2}}\right) \tag{73}$$

where $\kappa'_{12} \equiv \mathrm{d}\kappa_1/\mathrm{d}y_2$ and has the form (66) written in the statement of the theorem (we do not repeat it here for convenience). Analogously, the price of a European payer swaption is

$$-\sum_{i=m+1}^{n} c_i P(0,t_i) N\left(-\frac{\kappa_1(-\alpha_{i2}) + \alpha_{i1}}{\sqrt{1+[\kappa'_{12}(-\alpha_{i2})]^2}}\right). \tag{74}$$

Eqs. (73) and (74) are closed-form expressions which constitute the general formulas for the price of European swaptions in a HJM two-factor model.

6.3 A specific formulation of swaptions prices

Before concluding this part, we resume the assumptions made at the beginning of Sec. 4 as we would like to write down specific expressions for the α_i parameters, which, plugged into Eqs. (73) and (74), will provide a peculiar formulation of European swaptions prices in the context of the HJM two-factor model outlined above.
Given Eqs. (17), (26) (with $\hat{\sigma} \equiv \sigma$) and (31) for $n = 2$, we rewrite Eq. (60) as follows:

$$\nu^\alpha(t,u) = \int_t^u \sigma^\alpha(t,s)\mathrm{d}s = \sum_{\beta=1}^{2} h_t^{\alpha,\beta} \int_t^u e^{-\lambda^\beta(s-t)}\mathrm{d}s\,, \quad \alpha = 1,2, \tag{75}$$

which, performing the integral and substituting matrices (27) and (28), can be rewritten as

$$\nu^\alpha(t,u) = \sqrt{V_s}\frac{\sigma_1}{\lambda_1}\left(1-e^{-\lambda^0(u-s)}\right)\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{V_s}\frac{\sigma_2}{\lambda_2}\left(1-e^{-\lambda^1(u-s)}\right)\begin{pmatrix}\rho\\ \sqrt{1-\rho^2}\end{pmatrix}, \quad \alpha = 1,2, \tag{76}$$

where, for simplicity of notation, we have set $\sigma_1 \equiv \sigma^{11}, \sigma_2 \equiv \sigma^{22}$ and $\lambda_1 \equiv \lambda^1, \lambda_2 \equiv \lambda^2$. Moreover, we observe that ν^α is such that

$$\nu^\alpha(s,t_2) - \nu^\alpha(s,t_1) = \sum_{\beta=1}^{2} \frac{h_s^{\alpha,\beta}}{\lambda^\beta} e^{\lambda^\beta s}\left[e^{-\lambda^\beta t_1} - e^{-\lambda^\beta t_2}\right] = \sum_{\beta=1}^{2} g^{\alpha,\beta}(s) f^\beta(t_1,t_2)\,, \quad \alpha = 1,2, \tag{77}$$

which corresponds to condition (H) with $g^{\alpha,\beta}(s) \equiv h_s^{\alpha,\beta}e^{\lambda^\beta s}$ and $f^\beta(t_1,t_2) \equiv (e^{-\lambda^\beta t_1} - e^{-\lambda^\beta t_2})/\lambda^\beta$.
Now, starting from Eq. (67) in two dimensions:

$$\alpha_i^2 = \sum_{\alpha=1}^{2} \int_0^{t_m} (\nu^\alpha(s,t_i) - \nu^\alpha(s,t_m))^2 \mathrm{d}s\,, \tag{78}$$

which represents the square norm of vector $\alpha_i = (\alpha_{i1}, \alpha_{i2})$, $||\alpha_i||^2 = \sum_{k=1}^{2} \alpha_{ik}^2$, we are able to compute the quantities α_{i1}^2 and α_{i2}^2 as functions of the corresponding elements of vector ν^α:

$$\alpha_{i1}^2 = \int_0^{t_m} \left(\nu^1(s,t_i) - \nu^1(s,t_m)\right)^2 \mathrm{d}s \tag{79}$$

and

$$\alpha_{i2}^2 = \int_0^{t_m} \big(\nu^2(s,t_i) - \nu^2(s,t_m)\big)^2 \mathrm{d}s \, . \tag{80}$$

By using Eq. (76) separately for the two components, in combination with Eq. (30), and solving the integral, after some steps we get

$$\begin{aligned}\alpha_{i1}^2 = \frac{\sigma_1^2}{\lambda_1^2} F\big(2\lambda_1, \{\theta, V_0, k\}, t_m\big) G^2\big(\lambda_1, t_i; t_m\big) + \rho^2 \frac{\sigma_2^2}{\lambda_2^2} F\big(2\lambda_2, \{\theta, V_0, k\}, t_m\big) G^2\big(\lambda_2, t_i; t_m\big) \\ + \frac{2\rho\sigma_1\sigma_2}{\lambda_1\lambda_2} F\big(\lambda_1 + \lambda_2, \{\theta, V_0, k\}, t_m\big) G\big(\lambda_1, t_i; t_m\big) G\big(\lambda_2, t_i; t_m\big)\end{aligned} \tag{81}$$

and

$$\alpha_{i2}^2 = (1-\rho^2) \frac{\sigma_2^2}{\lambda_2^2} F\big(2\lambda_2, \{\theta, V_0, k\}, t_m\big) G^2\big(\lambda_2, t_i; t_m\big) \, , \tag{82}$$

where we have defined

$$F\big(\lambda, \{\theta, V_0, k\}, t\big) \equiv \frac{\theta}{\lambda}\Big(1 - e^{-\lambda t}\Big) - \frac{\theta - V_0}{\lambda - k}\Big(e^{-kt} - e^{-\lambda t}\Big) \tag{83}$$

and

$$G\big(\lambda, t_i; t\big) \equiv 1 - e^{-\lambda(t_i - t)} \, . \tag{84}$$

Moreover, denoting the quadratic terms by

$$Y_{kk}^i \equiv \frac{\sigma_k^2}{\lambda_k^2} F\big(2\lambda_k, \{\theta, V_0, k\}, t_m\big) G^2\big(\lambda_k, t_i; t_m\big) \, , \quad k = 1, 2 \tag{85}$$

and the mixed term by

$$Y_{12}^i \equiv \frac{\sigma_1\sigma_2}{\lambda_1\lambda_2} F\big(\lambda_1 + \lambda_2, \{\theta, V_0, k\}, t_m\big) G\big(\lambda_1, t_i; t_m\big) G\big(\lambda_2, t_i; t_m\big) \, , \tag{86}$$

and extracting the square roots of Eqs. (81) and (82), we obtain the following compact expressions for the two components of α_i:

$$\alpha_{i1} = \sqrt{Y_{11}^i + \rho^2 Y_{22}^i + 2\rho Y_{12}^i} \tag{87}$$

and

$$\alpha_{i2} = \sqrt{(1-\rho^2) Y_{22}^i} \, . \tag{88}$$

Just note that, since in Eq. (75) the correlation is inserted within the σ's, as we have assumed in Section 6.1, the components of ν^α and hence of α_i in Eq. (78) turn out to be mixed together. Therefore, it is not possible in general to express the elements of α_i as functions of their respective volatilities, contrary to what would happen in the scalar case. In fact, from Eqs. (87) and (88) we observe that, while α_{i2} depends only on σ_2, through the quadratic term Y_{22}^i, α_{i1} instead depends on both σ_1 and σ_2, given the presence of both the quadratic terms Y_{11}^i and Y_{22}^i and of the mixed term Y_{12}^i.

As we were saying at the beginning of this section, these final expressions for α_{i1} and α_{i2} can be substituted in Eqs. (73) and (74) to get a specific formulation of European swaptions prices in a HJM two-factor model.

7 Conclusions

In this paper, we presented an extension of the classic HJM framework, built in order to generate the dynamics of the extended forward term rates, which are equivalent to the FMM ones. This is precisely accomplished by matching the FMM dynamics using a parsimonious Markovian HJM model with separable volatility parameters. A model built in this way clearly differs from the older model of Moreni and Pallavicini based on IBOR rates and producing a multiple yield-curve dynamics from a single family of Markov processes, as it consists of a single-curve framework where all the structures are generated starting from a single rate, the risk-free rate. This model proved to be very suitable for evaluating vanilla derivatives on the new Overnight interest rate benchmarks that have replaced the IBORs. Indeed, by adopting a specific realization of it based on a deterministic form of the volatility, we had been able to write down the (exact) risk-free forward rate and the (approximated) risk-free swap rate dynamics, from which we directly obtained the final pricing Black-type formulas for the corresponding RFR derivatives (caps and swaptions). Moreover, by restricting our HJM model to the two-dimensional case, and following the analysis conducted by Henrard ([4], [5]), we could formulate explicit expressions for the prices of European, payer and receiver, swaptions.

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